MDS Axiomatic Foundations
(Young & Hamer, Chapter 4)

Metric Spaces
A space is said to be “a metric space” if it satisfies the following two axioms:

1. Non-Degeneracy: $d_{ij}=0$ iff $i=j$
2. Triangular Inequality (second version): $d_{jk} \leq d_{ij} + d_{ik}$. (The axioms about non-negativity and symmetry are implied by this axiom.)

Metric Data Representations
Observations must have certain properties to be representable by a metric space. Beals, Krantz & Tversky show that these properties include:

1. All pairs of identical stimuli are equally dissimilar and are less dissimilar than any pair of nonidentical stimuli.
2. The data must be symmetric
3. There must be an infinitessimally distinct pair of stimuli (and, therefore, an infinite number of stimuli).
4. There must be no “holes” in the stimuli.
5. ? I don’t understand the empirical implications of this one.
6. Cosegmentality – ?

Comment on Metric Data Representations:
They don’t seem to have much practical use empirically.

Vector Spaces & Norms
(Young & Hamer, Chap 4)

The r-dimensional vector space is a genralization of the concepts of line (1d), plane (2D) and "every-day space" (3D).

Definition:
An r-dimensional vector space is the set of all possible vectors, where a vector in an r-dimensional space is an ordered r-tuple of real numbers (v_{i1}, v_{i2}, ... , v_{ir}).

Operations in r-space:
Three operations are defined on vectors in r-dimensional vector space:

1. **Addition**: The sum of vectors is the vector whose elements are the sum of corresponding elements in the vectors.
2. **Subtraction**: The difference of vectors is the vector whose elements are the difference of corresponding elements in the vectors.
3. **Scalar Multiplication**: The product of a scalar and a vector is the vector having elements such that each element is the scalar times the original vector's element.

Properties of these operations: (see pages 79-80).

Geometry of these operations: (see pages 80-81).

Unweighted Distance Models
(Young & Hamer, Chapter 5)

The Minkowski Model:

$$d_{ij} = \left[\sum_{a=1}^{r} \left| x_{ia} - x_{ja} \right|^{p} \right]^{1/p} \quad (p \geq 1), x_i \neq x_j$$

Three special cases:

1. Euclidean Space: p=2
2. City-Block Space: p=1
3. Dominance Space: p=∞

Unweighted MDS Analyses

1. CMDS: One square data matrix.
2. RMDS: Many square data matrices
3. CMDU (Unfolding): One rectangular data matrix
4. RMDU: Many rectangular data matrices

Minkowski Spaces are Metrics
(their distances satisfy the metric axioms).

1. $d_{ii} = 0$ & $d_{ij} \geq 0$ (iff $x_i \neq x_j$)
2. Triangular inequality can be prooven to be true.
3. Non-negativity holds since we use the positive p-th root.
4. Symmetry holds since reversing i and j in the equation has no effect.

Algebraic Properties of Unweighted Distance Models

Similarity Transformations
Transformations which produce "similar" spaces, i.e., ones in which all distances between all pairs of points in one space are in ratio s to all distances between all pairs of points in the other space.

A similarity transformation is a restricted form of a general linear (affine) transformation.

The following operations are permitted in Euclidean Space:

1. Orthogonal Rotation
2. Translation
3. Permutation
4. Reflection
5. Central Dilation

The following are not permitted for similarity transformations, but are for general liner (affine) transformations:

1. Shear
2. Stretch

M-Similarity Transformations (Similarity Transformations in non-Euclidean, Minkowski spaces):

1. Includes: Translations, Permutation, Reflection, Central Dilation
2. Excludes: Orthogonal Rotation, Shear, Stretch

Geometric Properties of Unweighted Distance Models

See Young and Hamer figures 5.1-5.11, pp 94-106

Scalar Products

Scalar Products (also called "cross products" or "outer products") of a matrix **X** are simply defined as **B=XX'**

It is possible to convert a matrix **D** of Euclidean distances, which has coordinates **X,** into a matrix of scalar products **B** such that **B=XX'.**

This is done by "double centering" D^2, the matrix containing squared Euclidean distances, to obtain **B**. Then **X** is determined by performing a singular value decomposition of **B** into **B=UVU'.** Then $\mathbf{X=UV^{1/2}}$.

"Double centering" means you subtract the row and column means, and add back in the grand mean.